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A REVISION OF COUPLED MODE THEORY FOR IRREGULAR ACOUSTIC WAVEGUIDES

Ronald F. Pannatoni

DR. RONALD F. PANNATONI
120 Mark Dowdle Road
Franklin, North Carolina 28734

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800 North Quincy Street
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DCASMA ATLANTA
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We use the eigenfunction expansions to convert the reduced wave equation and the conditions at the boundaries of the waveguide into a system of first-order, ordinary differential equations. These coupling equations are similar to Shevchenko's equations for coupling between local modes in the waveguide [Sov. Phys. Acoust. 7 (1962) pp. 392-397, eqs. (16-18)].

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A revision of coupled mode theory for irregular acoustic waveguides

RONALD F. PANNATONI
Franklin, North Carolina, USA
August 8, 1988

1. INTRODUCTION

This note concerns new eigenfunction expansions for the fields in an irregular acoustic waveguide. The expansions are associated with a non-selfadjoint eigenvalue problem that has the eigenvalue in some of the boundary conditions. The waveguide consists of two layers of fluid that are separated by a frictionless free interface. The field expansions converge uniformly on both sides of the interface even if the fluid density is discontinuous at the interface.

We use the eigenfunction expansions to convert the reduced wave equation and the conditions at the boundaries of the waveguide into a system of first-order, ordinary differential equations. These coupling equations are similar to Shevchenko's equations for coupling between local modes in the waveguide [*Sov. Phys. Acoust.* 7(1962) pp. 392–397, eqs. (16–18)].

The eigenvalue problem and the field expansions are described in the next section. The coupling equations are stated in the last section.

2. THE EIGENFUNCTION EXPANSIONS

A. Notation and assumptions. The discussion in this note is limited to two-dimensional problems. The real variables x and z denote, respectively, the coordinates of horizontal and vertical position in the waveguide. The following notation is used for the indicated limits and jump discontinuities of an arbitrary function $F(x,z)$:

$$F(x,z+0) = \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} F(x,z+\epsilon), \tag{1}$$

$$F(x,z-0) = \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} F(x,z-\epsilon), \tag{2}$$

$$\underset{(x,z)}{\mathrm{jump}}[F] = F(x,z+0) - F(x,z-0). \tag{3}$$

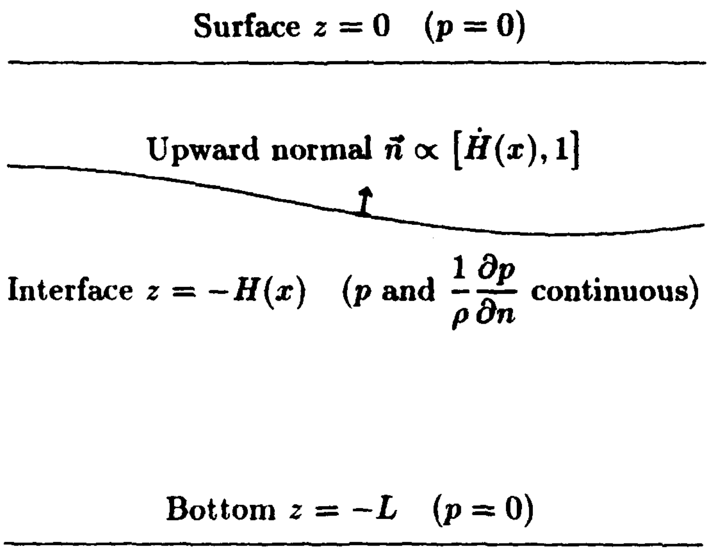

Fig. 1. Geometry of the waveguide and conditions at the boundaries.

A representative section of the waveguide is sketched in Figure 1. The surface boundary coincides with the horizontal line $z = 0$, and the bottom boundary coincides with the horizontal line $z = -L$. The interface between layers in the waveguide is described by the curve $z = -H(x)$. It is assumed that $0 < H(x) < L$ for all x and that the function $H(x)$ has a continuous second derivative.

The local sound speed $c(x, z)$ is continuous in each layer, and the local mass density $\rho(x, z)$ is continuously differentiable in each layer. These quantities may be discontinuous at the interface, but it is assumed that each of the four limits $c(x, -H(x) \pm 0)$ and $\rho(x, -H(x) \pm 0)$ exists and is positive.

Sound in the waveguide is assumed to depend harmonically on time with angular frequency ω. The acoustic pressure at time t can be expressed as the real part of the product $p(x, z)\exp(-i\omega t)$, where the complex pressure $p(x, z)$ solves a reduced wave equation in two dimensions: if $-L < z < -H(x)$ or $-H(x) < z < 0$ then

$$\rho(x,z)\frac{\partial}{\partial x}\left(\frac{1}{\rho(x,z)}\frac{\partial}{\partial x}p(x,z)\right) + \rho(x,z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(x,z)}\frac{\partial}{\partial z}p(x,z)\right) + \frac{\omega^2}{c(x,z)^2}p(x,z) = S(x,z). \tag{4}$$

The forcing term $S(x, z)$ in eq.(4) represents distributed sources of sound inside the waveguide.

The acoustic pressure vanishes at the external boundaries of the waveguide, and it is continuous at the interface:

$$p(x, -L) = 0, \tag{5}$$

$$p(x, 0) = 0, \tag{6}$$

$$\operatorname*{jump}_{(x, -H(x))} [p] = 0. \tag{7}$$

The fluid velocity associated with sound in the waveguide is proportional to the acoustic pressure gradient. The components of velocity in the horizontal and vertical directions can be expressed, respectively, as the real parts of the products $u(x, z)\exp(-i\omega t)$ and $w(x, z)\exp(-i\omega t)$, where the complex velocity components $u(x, z)$ and $w(x, z)$ are

$$u(x, z) = \frac{1}{i\omega\,\rho(x, z)} \frac{\partial}{\partial x} p(x, z), \tag{8}$$

$$w(x, z) = \frac{1}{i\omega\,\rho(x, z)} \frac{\partial}{\partial z} p(x, z). \tag{9}$$

At the interface the component of fluid velocity normal to the interface is continuous. This requirement can be expressed in terms of the complex velocity components as

$$\operatorname*{jump}_{(x, -H(x))} [w + \dot{H}(x)\, u] = 0, \tag{10}$$

where $\dot{H}(x) = (d/dx)H(x)$.

B. The eigenvalue problem. At each position x the following equations determine countably infinite sets of eigenvalues $k_n(x)$ and eigenfunctions $\phi_n(x, z)$:

(11)

$$\rho(x, z) \frac{\partial}{\partial z}\left(\frac{1}{\rho(x, z)} \frac{\partial}{\partial z} \phi_n(x, z)\right) + \left(\frac{\omega^2}{c(x, z)^2} - k_n(x)^2\right) \phi_n(x, z) = 0$$

$$\text{if } -L < z < -H(x) \text{ or } -H(x) < z < 0,$$

$$(12) \qquad \phi_n(x,-L) = 0,$$

$$(13) \qquad \phi_n(x,0) = 0,$$

$$(14) \qquad \underset{(x,-H(x))}{\text{jump}} \left[\dot{H}(x) \frac{\partial}{\partial z} \phi_n - i\, k_n(x)\, \phi_n \right] = 0,$$

$$(15) \qquad \underset{(x,-H(x))}{\text{jump}} \left[\frac{1}{\rho} \left(\frac{\partial}{\partial z} \phi_n + \dot{H}(x)\, i\, k_n(x)\, \phi_n \right) \right] = 0.$$

The eigenfunctions are assumed to be normalized in such a way that the mixed partial derivatives $(\partial^2/\partial x\, \partial z)\phi_n(x,0)$ exist at the surface.

For every x all but finitely many eigenvalues $k_n(x)$ are purely imaginary. The indices n of the eigenvalues run through the integers $0, \pm 1, \pm 2, \ldots$, and they can be assigned to the eigenvalues in such a way that for some integer $N(x)$ the following statements are true:

(a) if $|n| > N(x)$ then $k_n(x)$ is imaginary and $|k_n(x)| = O(|n|)$;
(b) if $n > N(x)$ then $-i\, k_n(x)$ is positive and increases in value as n increases;
(c) if $n < -N(x)$ then $-i\, k_n(x)$ is negative and decreases in value as n decreases.

C. Acoustic field expansions. For each index n define

$$(16) \quad a_n(x) = \int_{-L}^{0} \left(k_n(x)\, p(x,z) - i \frac{\partial}{\partial x} p(x,z) \right) \frac{1}{\rho(x,z)}\, \phi_n(x,z)\, dz + \frac{1}{k_n(x)}\, p(x,-H(x)) \underset{(x,-H(x))}{\text{jump}} \left[\frac{1}{\rho} \frac{\partial}{\partial z} \phi_n \right],$$

(17)
$$d_n(x) = 2\, k_n(x) \int_{-L}^{0} \frac{1}{\rho(x,z)}\, \phi_n(x,z)^2 dz + \frac{1}{k_n(x)} \underset{(x,-H(x))}{\text{jump}} \left[\phi_n \frac{1}{\rho} \frac{\partial}{\partial z} \phi_n \right].$$

The following expansions of the complex acoustic fields converge uniformly in the intervals $-L < z < -H(x)$ and $-H(x) < z < 0$:

$$(18) \qquad p(x,z) = \sum_{n=-\infty}^{\infty} \frac{a_n(x)}{d_n(x)}\, \phi_n(x,z),$$

(19) $$u(x,z) = \frac{1}{i\omega\,\rho(x,z)} \sum_{n=-\infty}^{\infty} \frac{a_n(x)}{d_n(x)}\, i\, k_n(x)\, \phi_n(x,z),$$

(20) $$w(x,z) = \frac{1}{i\omega\,\rho(x,z)} \sum_{n=-\infty}^{\infty} \frac{a_n(x)}{d_n(x)} \frac{\partial}{\partial z}\phi_n(x,z).$$

In particular, expansions (19) and (20) do not exhibit Gibb's phenomenon at $z = -H(x)$ when the velocity components $u(x,z)$ and $w(x,z)$ are discontinuous at the interface.

3. The coupling equations

The functions $a_n(x)$ that appear as coefficients in the acoustic field expansions solve an infinite system of first-order, ordinary differential equations. Each equation in this system takes the form

(21) $$\frac{d}{dx}a_n(x) = i\,k_n(x)\,a_n(x) - \sum_{m=-\infty}^{\infty} \frac{a_m(x)}{d_m(x)} c_{mn}(x) - i\,s_n(x),$$

(for $n = 0, \pm 1, \pm 2, \dots$)

where the forcing term $s_n(x)$ is a moment of the source distribution:

(22) $$s_n(x) = \int_{-L}^{0} S(x,z) \frac{1}{\rho(x,z)}\, \phi_n(x,z)\, dz.$$

It is possible to express the general coupling coefficient $c_{mn}(x)$ in terms of several factors to which physical meaning can be attributed. First, define the following quantities:

(23) $$L_{mn}(x,z) = \frac{\partial}{\partial z}\phi_m(x,z)\frac{\partial}{\partial z}\phi_n(x,z) + \left(k_m(x)\,k_n(x) - \frac{\omega^2}{c(x,z)^2}\right)\phi_m(x,z)\,\phi_n(x,z),$$

(24)
$$M_{mn}(x,z) = \frac{\partial}{\partial z}\phi_m(x,z)\frac{\partial}{\partial z}\phi_n(x,z) - k_m(x)\,k_n(x)\,\phi_m(x,z)\,\phi_n(x,z),$$

(25)
$$N_{mn}(x,z) = \frac{\partial}{\partial z}\phi_m(x,z)\frac{\partial}{\partial z}\phi_n(x,z) + k_m(x)\,k_n(x)\,\phi_m(x,z)\,\phi_n(x,z) + \dot{H}(x)\, i\left(k_m(x)\,\phi_m(x,z)\frac{\partial}{\partial z}\phi_n(x,z) + \frac{\partial}{\partial z}\phi_m(x,z)\,k_n(x)\,\phi_n(x,z)\right),$$

$$\text{(26)} \qquad \underset{(x,z)}{\mathrm{D}_{\|}}[1/\rho] = \frac{\partial}{\partial x}\left(\frac{1}{\rho(x,z)}\right) - \dot{H}(x)\frac{\partial}{\partial z}\left(\frac{1}{\rho(x,z)}\right).$$

Use these quantities to construct the following factors:

$$\text{(27)} \quad A_{mn}(x) = \int_{-L}^{0} \frac{\partial}{\partial x}\left(\frac{\omega^2}{c(x,z)^2}\right)\frac{1}{\rho(x,z)}\phi_m(x,z)\,\phi_n(x,z)\,dz - \int_{-L}^{0}\frac{\partial}{\partial x}\left(\frac{1}{\rho(x,z)}\right)L_{mn}(x,z)\,dz,$$

$$\text{(28)} \qquad I_{mn}(x) = \dot{H}(x) \underset{(x,-H(x))}{\text{jump}} \left[\frac{\omega^2}{c^2\rho}\phi_m\phi_n\right],$$

$$\text{(29)} \qquad J_{mn}(x) = \dot{H}(x) \underset{(x,-H(x))}{\text{jump}} [M_{mn}/\rho],$$

$$\text{(30)} \quad K_{mn}(x) = -\frac{\ddot{H}(x)\,i}{1+\dot{H}(x)^2}\underset{(x,-H(x))}{\text{jump}}[M_{mn}/\rho] + \frac{\dot{H}(x)\,i}{1+\dot{H}(x)^2}\underset{(x,-H(x))}{\text{jump}}\left[N_{mn}\,\mathrm{D}_{\|}[1/\rho]\right],$$

where $\ddot{H}(x) = (d^2/dx^2)H(x)$ and $m, n = 0, \pm 1, \pm 2, \ldots$.

The factor $A_{mn}(x)$ is associated with lateral variations in material properties of the fluids inside the waveguide. The factors $I_{mn}(x)$, $J_{mn}(x)$ and $K_{mn}(x)$ are associated with conditions at the interface. These factors can differ from zero only if material properties of the fluids are discontinuous at the interface, and if the slope or the curvature of the interface does not vanish. The following theorem states the relation of these factors to the general coupling coefficient $c_{mn}(x)$.

THEOREM. *If $m \neq n$ then*

$$\text{(31)} \quad c_{mn}(x) = \frac{1}{k_m(x)-k_n(x)}\left(A_{mn}(x) + I_{mn}(x) + \frac{k_m(x)}{k_n(x)}J_{mn}(x) + \frac{1}{k_n(x)}K_{mn}(x)\right).$$

However, if $m = n$ then

$$\text{(32)} \qquad c_{nn}(x) = -\frac{1}{2}\dot{d}_n(x) + \frac{1}{k_n(x)}J_{nn}(x) + \frac{1}{2\,k_n(x)^2}K_{nn}(x),$$

where $\dot{d}_n(x) = (d/dx)d_n(x)$.